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LETTER TO THE EDITOR

Density of state in a complex random matrix theory with external source

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Abstract. The density of state for a complex $N \times N$ random matrix coupled to an external deterministic source is considered for a finite N, and a compact expression in an integral representation is obtained.

The random matrix theory, in which the eigenvalues of random matrix are complex, may find some applications. For example, the two-dimensional electron systems under a strong magnetic field [1] or the study of a neural network [2] is similar to the random matrix theory.

A long time ago, Ginibre [3] considered the complex random matrix theory and obtained a density of state $\rho(z)$ ($z = x + \mathrm{i}y$); inside the circle in complex plane, the density of states $\rho(z)$ becomes uniformly flat and is vanishing outside the circle. This is a generalization of Wigner's semicircle law in the large N limit for the complex case.

The random matrix theory with an external source was recently investigated [4–16]. The external source is a deterministic, and non-random matrix, coupled to a random matrix. It has been discussed for a Hermitian random matrix [4–8] and for a chiral case [9]. Feinberg and Zee [10] studied the complex random matrix with an external source in the large N limit. The asymmetric random matrix with external source has also been studied [10, 11]. In the large N limit, the boundary of the density of state in the complex plane may be obtained by several methods. However, the expression for the density of state, in a finite N, is much harder for the external source problem. It is known that there appear interesting transitions of opening a gap by tuning the external source [5, 13]. It may be crucial to obtain an exact expression for the density of state in a finite N for such problems.

In this letter, we study a complex random matrix which couples to an external source matrix. We generalize the previous works for the real eigenvalues [6, 9] to this complex eigenvalue case. The density of state $\rho(z)$ for the complex eigenvalues z is given by

$$\rho(z) = \frac{1}{N}\left\langle \sum_{i=1}^{N} \delta(x - \mathrm{Re}\,\lambda_i)\delta(y - \mathrm{Im}\,\lambda_i) \right\rangle \tag{1}$$

where λ_i is an eigenvalue of a complex matrix M, which couples to the external source matrix A through the following probability distribution for this case,

$$P_A(M) = \frac{1}{Z_A} \mathrm{e}^{-N\,\mathrm{tr}\,M^\dagger M + N\,\mathrm{tr}(M^\dagger A + A^\dagger M)}. \tag{2}$$

It has obtained by Ginibre [3] for a finite N, and $A = 0$ as

$$\rho(z) = \frac{1}{\pi}\sum_{n=0}^{N-1}\frac{N^n|z|^{2n}}{n!}\mathrm{e}^{-N|z|^2} \tag{3}$$

where we write the result in which the radius of the disk is unity in the large N limit as a normalization. To evaluate the density of state (1), it is useful to consider a chiral Hamiltonian,

$$H = \begin{pmatrix} 0 & M^\dagger \\ M & 0 \end{pmatrix} + \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix} \tag{4}$$

where M is a complex matrix. We denote the density of state of this chiral Hamiltonian by $\rho_{ch}(\lambda)$,

$$\rho_{ch}(\lambda) = \frac{1}{2N}\langle \mathrm{tr}\,\delta(\lambda - H)\rangle \tag{5}$$

where the probability distribution is $P(H) = \frac{1}{Z}\exp[-N\,\mathrm{tr}\,M^\dagger M]$. Note that the eigenvalues of H are always real and appear in pairs of positive and negative values. Due to this chirality of the eigenvalues, the density of state $\rho_{ch}(\lambda)$ is equal to

$$\rho_{ch}(\lambda) = |\lambda|\tilde{\rho}(\lambda^2) \tag{6}$$

where

$$\tilde{\rho}(r) = \frac{1}{N}\langle \mathrm{tr}\,\delta(r - M^\dagger M)\rangle \tag{7}$$

in which the average distribution probability $P(M)$ is the same as $P_A(M)$ in (2).

As noticed by Feinberg and Zee [10], the density of state $\rho(z)$ in (1) is obtained from the expression of the density of state $\rho_{ch}(\lambda)$ by the shift of A in (2) as $A \to A - zI$. Using the well known expressions for the complex delta-function $\delta(z) = \delta(x)\delta(y)$, $z = x + \mathrm{i}y$,

$$\delta(z - z_0) = \frac{1}{\pi}\frac{\partial}{\partial z^*}\left(\frac{1}{z - z_0}\right) \tag{8}$$

$$\pi\delta(z) = \partial_z\partial_{z^*}\log(zz^*) \tag{9}$$

we have

$$\rho(z) = \frac{1}{\pi}\partial_z\partial_{z^*}\left\langle\frac{1}{N}\,\mathrm{tr}\log(z - M)(z^* - M^\dagger)\right\rangle. \tag{10}$$

Using a dispersion relation between the Green function and the density of state $\rho_{ch}(\lambda)$, we obtain

$$\begin{aligned}\rho(z) &= -\frac{2\mathrm{i}}{\pi}\int_0^\infty \mathrm{d}s\,\partial_z\partial_{z^*}\left(\int_{-\infty}^{\infty}\frac{\rho_{ch}(\lambda)}{\mathrm{i}s - \lambda}\,\mathrm{d}\lambda\right)\\ &= -\frac{4}{\pi}\partial_z\partial_{z^*}\int_0^\infty \mathrm{d}s\int_0^\infty \mathrm{d}\lambda\,\frac{\lambda s}{\lambda^2 + s^2}\tilde{\rho}(\lambda^2)\end{aligned} \tag{11}$$

in which the external source A is shifted as $A = \mathrm{diag}(|a_1 - z|\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, |a_N - z|\mathrm{e}^{\mathrm{i}\theta_N})$. In the large N limit, this $\tilde{\rho}(\lambda^2)$ was obtained by a diagrammatic analysis, and the density of state $\rho(z)$ was obtained by this procedure [10]. We consider here the finite N case, not in the large N limit, by calculating the chiral $\rho_{ch}(\lambda)$ with an external source through the Itzykson–Zuber integral [17].

A complex matrix M is decomposed as

$$M = UXV \tag{12}$$

where U and V are unitary matrices and X is a diagonal matrix. Since the number of real variables is $2N^2$ for M, N^2 for U,V and $2N$ for X, we have $2N$ conditions on U, V and X. It is possible to put the condition that the diagonal element of V is real, and $X = \mathrm{diag}(x_1, \ldots, x_N)$, x_i is real, $x_i \geqslant 0$. Note that x_i is not an eigenvalue of M, but x_i^2 is an eigenvalue of $M^\dagger M$. x_i is called a singular value of M.

The Itzykson–Zuber integral for this case is known [18–20],

$$\int \mathrm{d}U\,\mathrm{d}V\,\mathrm{e}^{\mathrm{Re}\,(\mathrm{tr}\,UXVY)} = \frac{(2\pi)^{N^2}}{N!}\frac{\det[I_0(x_i y_j)]}{\Delta(x^2)\Delta(y^2)} \tag{13}$$

where $Y = \mathrm{diag}(y_1, \ldots, y_N)$ and $\Delta(x^2) = \prod_{i<j}(x_i^2 - x_j^2)$, which is a Van der Monde determinant. This Itzykson–Zuber integral is obtained by applying a Laplacian of M to (13). This Laplacian reduces to the diagonal one, and (13) is a zonal spherical function.

The external source $A = \mathrm{diag}(|a_1 - z|\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, |a_N - z|\mathrm{e}^{\mathrm{i}\theta_N})$ is decomposed as $A = Y\tilde{U}$, $\tilde{U} = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_N})$ and $Y = \mathrm{diag}(|a_1 - z|, \ldots, |a_N - z|)$. This phase unitary matrix $\tilde{U}$ can be absorbed in U. Thus, we have a diagonal matrix element $y_i = |a_i - z|$ in Y.

Using the contour representation method by Kazakov [21], and taking the same procedure as used by Brézin, Hikami and Zee (BHZ) [9], we evaluate the evolution operator $U_A(t)$, which is a Fourier transform of the density of state $\tilde{\rho}(\lambda)$ in (7), by noting that x_i^2 is an eigenvalue of $M^\dagger M$,

$$\tilde{\rho}(\lambda) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi}\mathrm{e}^{-\mathrm{i}t\lambda}U_A(t) \tag{14}$$

$$U_A(t) = \frac{1}{N}\langle \mathrm{tr}\,\mathrm{e}^{\mathrm{i}tM^\dagger M}\rangle = \frac{1}{NZ_A}\sum_{\alpha}^{N}\int_0^{\infty}\prod_{i=1}^{N}\mathrm{d}x_i\,x_i\frac{\Delta(x^2)}{\Delta(y^2)}\det[I_0(2Nx_i y_j)]\mathrm{e}^{-N\sum x_i^2 + \mathrm{i}tx_\alpha}. \tag{15}$$

The coefficient of (14) is not important since we normalize $U_A(t)$ as $U_A(0) = 1$. This expression is similar to the previous results [6, 8] except that we have a modified Bessel function $I_0(2Nx_i y_j)$ instead of $\mathrm{e}^{Nx_i y_j}$ as an element of the determinant. The modified Bessel function has an integral representation as

$$I_0(2\sqrt{a}) = \int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{2\pi}\mathrm{e}^{\mathrm{e}^{\mathrm{i}\theta} + a\mathrm{e}^{-\mathrm{i}\theta}}. \tag{16}$$

Keeping the notation $y_i^2 = |a_i - z|^2$, we find the integral representation for $U_A(t)$ as

$$U_A(t) = \frac{1}{N}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{2\pi}\int_0^{\infty}\mathrm{d}q\oint\frac{\mathrm{d}u}{2\pi\mathrm{i}}\frac{1}{(1 - \frac{\mathrm{i}t}{N} - u)(f - u\mathrm{e}^{\mathrm{i}\theta})}$$
$$\times\mathrm{e}^{-q + \mathrm{i}\theta + (1-u)\mathrm{e}^{\mathrm{i}\theta}}\prod_{i=1}^{N}\left(\frac{f - Ny_i^2}{u\mathrm{e}^{\mathrm{i}\theta} - Ny_i^2}\right) \tag{17}$$

where $f = (\frac{\mathrm{i}t + Nu}{N - \mathrm{i}t - Nu})q$. The contour integral over u is reduced to evaluation of the residue at the pole $u = Ny_i^2\mathrm{e}^{-\mathrm{i}\theta}$. The integration over q is intoduced for the absorption of a combinatorial factor $k!$, which appears in the x_i integral in (14).

One can easily find that when there is no external source $y_i = 0$, the expression $U_{A=0}(t)$ in (17) reduces to the result of BHZ [9]. The density of state $\tilde{\rho}(\lambda)$ is given by the shift of $t \to N(t + \mathrm{i}u)$,

$$\tilde{\rho}(\lambda) = \int_{-\infty}^{\infty}\frac{\mathrm{d}t}{2\pi}\oint\frac{\mathrm{d}u}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}v}{2\pi\mathrm{i}}\int_0^{\infty}\mathrm{d}q\,\frac{\mathrm{e}^{-q - \mathrm{i}Nt\lambda + Nu\lambda - v(1 - \frac{1}{u})}}{u(1 - \mathrm{i}t)(f - v)}\prod_i\left(\frac{f - Ny_i^2}{v - Ny_i^2}\right) \tag{18}$$

where $f = \mathrm{i}tq/(1-\mathrm{i}t)$, and we have made a change of variable $\mathrm{e}^{\mathrm{i}\theta} = v/u$. Now we consider the density of state $\rho(z)$ for the complex matrix through (11). We replace a factor as

$$\frac{1}{s^2+\lambda^2} = \int_0^\infty \mathrm{e}^{-\alpha(s^2+\lambda^2)}\,\mathrm{d}\alpha. \tag{19}$$

Inserting (18) into (11), and integrating over s, λ and t, we obtain by the change of variables, $u \to u + \alpha/N$, $q \to (1-u)q/u$, $\alpha \to N\alpha u$ and $\beta = \alpha/(\alpha+1)$,

$$\rho(z) = \frac{1}{\pi N}\partial_z\partial_{z^*}\left[\int_0^1 \mathrm{d}\beta \oint \frac{\mathrm{d}u}{2\pi\mathrm{i}} \oint \frac{\mathrm{d}v}{2\pi\mathrm{i}} \int_0^\infty \mathrm{d}q\, \frac{\mathrm{e}^{-\frac{1-u}{u}q - v + \frac{v}{u}(1-\beta)}}{\beta u^2(q-v)} \prod_{i=1}^N \left(\frac{q-Ny_i^2}{v-Ny_i^2}\right)\right] \tag{20}$$

where the contour of the integration of v is around Ny_i^2 and the contour of u is around $u=1$, which appears as a pole after the integration of q.

If $f(q)$ is a polynomial of q, we are able to prove that

$$\oint \frac{\mathrm{d}u}{2\pi\mathrm{i}} \int_0^\infty \mathrm{d}q\, \frac{1}{u^2}\mathrm{e}^{\frac{v(1-\beta)}{u}}\mathrm{e}^{-\frac{(1-u)q}{u}} f(q) = -\mathrm{e}^{v(1-\beta)} f(v(1-\beta)). \tag{21}$$

Thus, the integrations over q and u can be done, and we finally obtain by the shift $v \to Nv$,

$$\rho(z) = \frac{1}{\pi N}\partial_z\partial_{z^*}\left[\int_0^1 \mathrm{d}\beta \oint \frac{\mathrm{d}v}{2\pi\mathrm{i}} \frac{\mathrm{e}^{-\beta N v}}{\beta^2 v} \prod_{i=1}^N \left(1 - \frac{\beta v}{v - y_i^2}\right)\right] \tag{22}$$

where contours are taken around all y_i^2.

It is easy to write down the explicit form for small N. We have

$$\begin{aligned}
\rho(z) &= \frac{1}{\pi}\mathrm{e}^{-|a_1-z|^2} \qquad (N=1)\\
&= \frac{1}{\pi}\left[\mathrm{e}^{-2|a_1-z|^2} + \mathrm{e}^{-2|a_2-z|^2} - \frac{1}{4}\partial_z\partial_{z^*}\left(\frac{\mathrm{e}^{-2|a_1-z|^2} - \mathrm{e}^{-2|a_2-z|^2}}{|a_1-z|^2 - |a_2-z|^2}\right)\right] \qquad (N=2)\\
&= \frac{1}{\pi}\left[\sum_{i=1}^3 \mathrm{e}^{-3y_i^2} - \frac{1}{9}\partial_z\partial_{z^*}\oint \frac{\mathrm{d}v}{2\pi\mathrm{i}} \frac{\mathrm{e}^{-3v}(6v - 3\sum_{i=1}^3 y_i^2 - 1)}{\prod_{i=1}^3 (v - y_i^2)}\right] \qquad (N=3)
\end{aligned} \tag{23}$$

where $y_i^2 = (a_i - z)(a_i^* - z^*)$. It is also easy to see that, when we put $a_i = 0$, we obtain $y^2 = z^* z$, and by the differentiation for z and z^*, (22) becomes

$$\rho_N(z) = \frac{1}{\pi}\int_0^1 \mathrm{d}\beta \oint \frac{\mathrm{d}v}{2\pi\mathrm{i}} \left(Ny^2 v - \frac{1}{\beta}\right)\left(1 - \frac{\beta v}{v-1}\right)^N \mathrm{e}^{-\beta N v y^2}. \tag{24}$$

If we write Ny^2 by s^2, and put $I_N(s) = \rho_N(z) - \rho_{N-1}(z)$, we find $I_N(s) = \frac{s^{2(N-1)}}{(N-1)!}\mathrm{e}^{-s^2}/\pi$. This agrees with Ginibre's result (3). We can immediately obtain the boundary of the density of state from (22) by the saddle-point equation. Taking the derivative of the exponent in the large N limit by β, and putting $\beta = 1$, which is an endpoint of the integral, we have as a boundary curve $x^2 + y^2 = 1$ for the Ginibre case, and $x^4 + y^4 + 2x^2y^2 - 3x^2 + y^2 = 0$ for the case when the external source eigenvalues are $a_i = \pm 1$, $N/2$ times degenerated.

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